Necessary and sufficient conditions for robust D-symmetrizability of matrices

by

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Abstract. An algorithm for matrix D-symmetrizability verification is presented. Necessary and sufficient conditions for the D-symmetrizability of matrix convex combination are proved.

Key words: symmetric matrix, robust symmetrizability

1. Introduction. Necessary and sufficient conditions for a matrix to be symmetrizable through multiplication by a positive definite diagonal matrix (D-symmetrizability) are presented in Section 2. A more general problem (symmetrizability by positive definite matrix) has been considered by Taussky [5]. Fiedler et al [2], Johnson and da Silva [4], and Jameson et al [3] have presented some properties of D-symmetrizable matrices. The problem of whether a convex combination of two D-symmetrizable matrices is also D-symmetrizable is considered in Section 3. Necessary and sufficient conditions for robust D-symmetrizability have been obtained.

2. D-symmetrizable matrices.

DEFINITION 1. *Matrix* $A = [a_{ij}] \in \Re^{nxn}$ *is D-symmetrizable, if there exists a diagonal matrix*

(1) $$K = \mathrm{diag}(k_1, k_2, ..., k_n) \in \Re^{nxn}$$

such that $k_i > 0$ *(i=1,2,...,n) and*

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(2) $$KA = A^T K.$$

D-symmetribility of matrix $A = \lfloor a_{ij} \rfloor$ does not depend on elements a_{ii} where $i = 1,2,\ldots,n$. Equation (2) is satisfied if

(3) $$a_{ij} k_i = a_{ji} k_j$$

for $i = 1,2,\ldots,n-1$; $j = 2,3,\ldots,n$ and $i < j$. For a diagonal matrix *K*, equation (2) can be rewritten in the alternative matrix notation

(4) $$L(A)\, k = 0$$

where $k = [k_1, k_2, ..., k_n]^T \in \Re^n$ and

(5) $$L(A) = \begin{bmatrix} a_{12} & -a_{21} & 0 & 0 & \cdots & 0 & 0 \\ a_{13} & 0 & -a_{31} & 0 & \cdots & 0 & 0 \\ a_{14} & 0 & 0 & -a_{41} & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{1n} & 0 & 0 & 0 & \ldots & 0 & -a_{n1} \\ 0 & a_{23} & -a_{32} & 0 & \ldots & 0 & 0 \\ 0 & a_{24} & 0 & -a_{42} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & a_{2n} & 0 & 0 & \ldots & 0 & -a_{n2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & a_{n-1,n} & -a_{n,n-1} \end{bmatrix} \in \Re^{[n(n-1)/2]xn}.$$

By $\mathrm{rk}(L)$ let us denote the rank of matrix *L* and let

$$\mathrm{sign}\,(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

THEOREM 1. [1] *Matrix* $A = a_{ij} \in \Re^{nxn}$*, such that* $a_{ij} \neq 0$ *for* $i, j = 1,2,...,n$ *and* $i \neq j$*, is D-symmetrizable if and only if*

(6) $$\mathrm{rk}(L(A)) < n$$

and

(7) $$\mathrm{sign}(a_{ij}) = \mathrm{sign}(a_{ji}) \quad (i, j = 1,2,...,n).$$

EXAMPLE 1. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \Re^{5\times 5}$$

then $\mathrm{rk}(L(A)) = 4 < 5$, $\mathrm{sign}(a_{ij}) = \mathrm{sign}(a_{ji})$ for $i, j = 1,\ldots,5$ but matrix A is not D-symmetrizable because some $a_{ij} = 0$ for $i \neq j$ and equation $L(A)\, k = 0$ has the solution $k = [0, 0, 0, 1, 1]^T$.

Now we shall present an algorithm which will enable us to verify the D-symmetrizability of matrix $A = [a_{ij}] \in \Re^{n\times n}$ for the more general case where assumption $a_{ij} \neq 0$ for $i \neq j$ is not needed. Let us consider the sets $NJ = \varnothing$ and $NI = \{1, 2, ..., n\}$ where $\varnothing$ denotes the empty set.

ALGORITHM:

1) If all equations (7) hold for $i \neq j$, then go to point 2); in the opposite case matrix A is not D-symmetrizable and the verification is finished.
2) Substitute $k_1 = 1$, $NJ = \{1\}$, $NI = NI / \{1\}$.
3) If there exists $(i, j) \in NI \times NJ$ such that $a_{ij} \neq 0$ then calculate $k_i = k_j a_{ji} / a_{ij}$, $NJ = NJ \cup \{i\}$, $NI = NI / \{i\}$ and go to point 3).
4) If all $k_1, k_2, \ldots, k_n$ have been determined, then go to point 5). If there exists k_m which has not been calculated, take $k_m = 1$, $NJ = NJ \cup \{k_m\}$, $NI = NI / \{k_m\}$ and go to point 3).
5) Verify conditions (3). If all of them hold, matrix $A = \lfloor a_{ij} \rfloor$ is D-symmetrizable by matrix $K = diag(k_1, k_2, \ldots, k_n) > 0$. In the opposite case matrix $A = \lfloor a_{ij} \rfloor$ is not D-symmetrizable.

ALGORITHM MOTIVATION.

From (3) it follows that if matrix $A=[a_{ij}]\in\Re^{n\times n}$ is symmetrizable by matrix $K=\mathrm{diag}(k_1,k_2,\ldots,k_n)$ then $\mathrm{sign}(a_{ij})=\mathrm{sign}(a_{ji})$ for $i,j=1,2,\ldots,n$. Moreover, if $k=[k_1,k_2,\ldots k_n]^T\in\Re^n$ is a solution of equation (4), then $\tilde{k}=[1,k_2/k_1,k_3/k_1,\ldots,k_n/k_1]^T$ is a solution of (4), also. This motivates algorithm points 1) and 2). If in point 3) some k_m was not calculated, it means that k_m is independent of k_i (where $i\in NJ$), because e.g. $a_{mj}=0$ (for $j=1,2,\ldots,n$; $j\neq m$). In such a case we can take $k_m=1$. This justifies algorithm point 4).

3. Robust D-symmetrizability. It appears that there exist such D-symmetrizable matrices that each of their convex combinations is a D-symmetrizable matrix. We shall present the necessary and sufficient conditions for generating the set of D-symmetrizable matrices spanned by the two D-symmetrizable matrices *P* and *Q*.

DEFINITION 2. $\mathbf{S}$ *is a set of D-symmetrizable matrices if for every matrix* $A\in\mathbf{S}\subset\Re^{nxn}$ *there exists a diagonal positive definite matrix* K *such that* $KA=A^TK$.

Now, let us consider the set of matrices

(8) $$\mathbf{U}=\{\alpha P+(1-\alpha)Q:\alpha\in[0,1]\}$$

where $P=[p_{ij}]$, $Q=[q_{ij}]\in\Re^{nxn}$. The set $\mathbf{U}$ is a convex combination of matrices *P* and *Q*. Our task is to find properties of *P* and *Q* which yield necessary and sufficient conditions for the set $\mathbf{U}$ to be D-symmetrizable, i.e. $\mathbf{U}\subset\mathbf{S}$.

DEFINITION 3. *Matrices* $P,Q\in\Re^{nxn}$ *have consistent signs if*

(9) $$\mathrm{sign}(p_{ij})=\mathrm{sign}(p_{ji})=\mathrm{sign}(q_{ij})=\mathrm{sign}(q_{ji})\neq 0$$

for i,j=1,2,...,n and i≠j.

It is easy to notice that if matrices $P=[p_{ij}]$, $Q=[q_{ij}]\in\Re^{nxn}$ have consistent signs then for all $\alpha\in[0,1]$

(10) $$\operatorname{sign}\left(\alpha p_{ij} + (1-\alpha) q_{ij}\right) = \operatorname{sign}\left(\alpha p_{ji} + (1-\alpha) q_{ji}\right)$$

for i,j=1,2,...,n and i≠j.

For the linear combination of matrices $P = \left[p_{ij}\right], Q = \left[q_{ij}\right] \in \Re^{nxn}$ and $\alpha \in [0,1]$ let $L(\alpha P + (1-\alpha)Q) \in \Re^{[n(n-1)/2]xn}$ be the matrix defined by (5). Without loss of generality we can assume $n > 2$. Let us denote the minor

(11) $$L(\alpha)\begin{pmatrix} i_1, i_2, ..., i_n \\ 1, 2, ..., n \end{pmatrix}$$

of matrix $L(\alpha P + (1-\alpha)Q)$ taken from rows $i_1 < i_2 < \cdots < i_n$ and columns $1, 2, ..., n$. This minor is a polynomial of variable α

(12) $$L(\alpha)\begin{pmatrix} i_1, i_2, ..., i_n \\ 1, 2, \ldots, n \end{pmatrix} = a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1 \alpha + a_0 = f(\alpha)$$

and the degree of polynomial (12) satisfies the inequality condition $\deg(f(\alpha)) \le n$. Let us denote the set, which contains these polynomials, by

(13) $$\mathbf{M}(\alpha) = \left\{ L(\alpha)\begin{pmatrix} i_1, i_2, ..., i_n \\ 1,2,..., n \end{pmatrix} : 1 \le i_1 < i_2 < \cdots < i_n \le n(n-1)/2 \right\}.$$

LEMMA 1. *Inequality*

(14) $$\operatorname{rk}(L(\alpha P + (1-\alpha)Q)) < n$$

is satisfied for all $\alpha \in [0,1]$ *if and only if each polynomial*

(15) $$f(\alpha) = a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1 \alpha + a_0 \in \mathbf{M}(\alpha)$$

fulfils conditions

(16) $$a_n = a_{n-1} = \cdots = a_0 = 0.$$

P r o o f. If (14) is satisfied for all $\alpha \in [0,1]$ then $f(\alpha) = 0$ for every polynomial $f(\alpha) \in \mathbf{M}(\alpha)$ and for all $\alpha \in [0,1]$. This means that (16) must be fulfilled. The sufficient condition results from (16) and the definition of $\mathbf{M}(\alpha)$.

Now we shall prove necessary and sufficient condition for the D-symmetrizability of set $\mathbf{U} = \{\alpha P + (1-\alpha) Q : \alpha \in [0,1]\}$.

THEOREM 2. *If matrices* $P = [p_{ij}], Q = [q_{ij}] \in \Re^{nxn}$ *have consistent signs, then set* $\mathbf{U} = \{\alpha P + (1-\alpha) Q : \alpha \in [0,1]\}$ *is D-symmetrizable if and only if each polynomial* $f(\alpha) = a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1 \alpha + a_0 \in \mathbf{M}(\alpha)$ *fulfils (16).*

P r o o f. Let us take into consideration arbitrary $\alpha_0 \in [0,1]$ and matrix $\alpha_0 P + (1-\alpha_0) Q = [\alpha_0 p_{ij} + (1-\alpha_0) q_{ij}] \in \mathbf{U}$. From the assumption that matrices P and Q have consistent signs, and from (10) we obtain

(17) $$\operatorname{sign}\left(\alpha_0 p_{ij} + (1-\alpha_0) q_{ij}\right) = \operatorname{sign}\left(\alpha_0 p_{ji} + (1-\alpha_0) q_{ji}\right) \neq 0$$

for i,j=1,2,...,n and i≠j.

Sufficient condition. Let us assume, that condition (16) is satisfied for every $f(\alpha) = a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1 \alpha + a_0 \in M(\alpha)$. From that and, moreover, from Lemma 1 $\operatorname{rk}\left(L(\alpha_0 P + (1-\alpha_0) Q)\right) < n$ results. From this inequality and from (17), on the basis of Theorem 1, it follows that matrix $\alpha_0 P + (1-\alpha_0) Q$ is D-symmetrizable.

Necessary condition. From the assumption that matrix $\alpha P + (1-\alpha) Q$ is D-symmetrizable for every $\alpha \in [0,1]$ and from Theorem 1, $\operatorname{rk}\left(L(\alpha P + (1-\alpha) Q)\right) < n$ results. From that and from Lemma 1 it follows that condition (16) is satisfied for every $f(\alpha) = a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1 \alpha + a_0 \in M(\alpha)$.

EXAMPLE 2. Let us consider matrices

$$P = \begin{bmatrix} 0 & 3 & -3 \\ 2 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 4 & -4 \\ 4 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix}.$$

Matrices P and Q have consistent signs, according to Definition 4. For the considered example

$$L(\alpha P + (1-\alpha)Q) = \begin{bmatrix} 4-\alpha & 2\alpha - 4 & 0 \\ \alpha - 4 & 0 & 2-\alpha \\ 0 & 2 & -1 \end{bmatrix}$$

and $\det(L(\alpha P + (1-\alpha)Q) \equiv 0$, $M(\alpha) = \{\det(L(\alpha P + (1-\alpha)Q) \equiv 0\}$. From Theorem 2 it follows that matrix $\alpha P + (1-\alpha)Q$ is D-symmetrizable for each $\alpha \in [0,1]$. This means that matrix

$$\alpha P + (1-\alpha)Q = \begin{bmatrix} 0 & 4-\alpha & \alpha - 4 \\ 4 - 2\alpha & 0 & 2 \\ \alpha - 2 & 1 & 0 \end{bmatrix}$$

is symmetrizable by matrix

$$K = \operatorname{diag}\left(1, \frac{\alpha - 4}{2(\alpha - 2)}, \frac{\alpha - 4}{\alpha - 2}\right) > 0$$

for each $\alpha \in [0,1]$.

References

[1] S. Białas, *A generalization of the criterion that a quadratic form is positive definite,* Bull.Pol.Acad.Scien., 43(1) (1995) 337-400.

[2] M.Fiedler, C.R.Johnson, T.L.Markham, M.Neumann, *A trace inequality for M-matrices and the symmetrizability of a real matrix by a positive diagonal matrix*, Linear Algebra and Its Applications, 71 (1985) 81-94.

[3] A.Jameson, E.Kreindler, P.Lancaster, *Symmetric, positive semidefinite, and positive definite real solution of* $AX = XA^T$ *and* $AX = YB$, Linear Algebra and Its Applications, 160 (1992) 189-215.

[4] C.R.Johnson, J.A.Dias da Silva, *Symmetric matrices associated with a nonnegative matrix,* Circuits Systems Signal Process, 9 (1990) 171-180.

[5] O. Taussky, *Positive-definite matrices and their role in the study of the characteristic roots of general matrices,* Advan. Math., 2 (1968) 175-186.